ETMAG

CORONALECTURE 9

Linear independence – ctd.

Matrices

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Definition. (alternate definition od *span*)
Let $S \subseteq V$ (a sub**set**, not necessarily a sub**space**). Then by *span*(*S*) we denote the smallest subspace of V containing S.

We call span(S) the *subspace spanned by S.*

One advantage of this definition over the other one is it covers the case $S = \varnothing$ without branching.

Fact.
Let V(S) denote the set of all subspaces of V containing S. Then

$$span(S) = \bigcap_{T \in V(S)} T$$

Proof. It is enough to show that intersection of a collection of subspaces is a subspace of V and that is easy. (All contain Θ so intersection does, too, etc.)

Theorem.

The set $S=\{v_1,v_2, \dots ,v_n\}$ is linearly independent iff no vector from S is a linear combination of the others.

Proof. ($\Rightarrow$) Suppose one of the vectors is a linear combination of the others. Without loss of generality we may assume that v_n is the one, i.e. $v_n=a_1v_1+a_2v_2+ \dots .a_{n-1}v_{n-1}$. Then we may write $\Theta=a_1v_1+a_2v_2+ \dots .a_{n-1}v_{n-1}+(-1)v_n$. Since $-1 \neq 0$ the set $\{v_1,v_2, \dots ,v_n\}$ is linearly dependent.

($\Leftarrow$) Suppose now that $\{v_1,v_2, \dots ,v_n\}$ is linearly dependent, i.e. there exist coefficients $a_1,a_2, \dots ,a_n$, not all of them zeroes, such that $\Theta=a_1v_1+a_2v_2+ \dots .a_nv_n$. Again, without losing generality, we may assume that $a_n\neq0$ (we can always renumber the vectors so that the one with nonzero coefficient is the last). Since nonzero scalars are invertible, we have $v_n=(-a_1a_n^{-1})v_1 + (-a_2a_n^{-1})v_2 + \dots +(-a_{n-1}a_n^{-1})v_{n-1}$

Examples. (on linear independence)

Decide which sets are linearly independent:

1. $\{(1,0),(0,1)\}$ in $\mathbb{R}^2$ over $\mathbb{R}$

2. $\{(x,y),(2x,2y)\}$ in $\mathbb{R}^2$ over $\mathbb{R}$

3. $\{(1,2,1),(1,-2,1),(2,0,2)\}$ in $\mathbb{R}^3$ over $\mathbb{R}$

4. $\{1,x,x^2,\dots,x^n\}$ in $R_n[x]$ over $\mathbb{R}$

5. $\{\sin x,\cos x,x\}$ in $\mathbb{R}^{\mathbb{R}}$ over $\mathbb{R}$

6. $\left\{\{a,b\},\{a\},\varnothing\right\}$ in $2^{\{a,b,c\}}$ over $\mathbb{Z}_2$

Example 5.

$\{\sin x, \cos x, x\}$ in $\mathbb{R}^{\mathbb{R}}$ over $\mathbb{R}$

Solution. Consider $a \sin x + b \cos x + c\, x = \Theta$. The golden question is *what the hell is* Θ (*zero vector*) *in* $\mathbb{R}^{\mathbb{R}}$*?* Obviously the constant zero function, $\Theta(x)=0$ for every x. Hence our condition means: $(\forall x \in \mathbb{R})\ a \sin x + b \cos x + c\, x = \Theta(x) = 0$. This means whatever number we replace x with the equality hold. Try $x=0$. We get $a0 + b1 + c0 = 0$, which means $b=0$. Knowing $b=0$, try $x=\pi$. This gives us $a\, 0 + 0(-1) + c\pi = 0$, so $c=0$. Putting $x = \frac{\pi}{2}$ we get $a1 = 0$, $a=0$.

Theorem.

Suppose V is a vector space, dimV=n, n>0 and S⊆V. Then

1. If |S|=n and S is linearly independent then S is a basis for V
2. If |S|=n and span(S)=V then S is a basis for V
3. If S is linearly independent then |S|≤n
4. If S spans V then S contains a basis of V
5. S is a basis of V iff S is a maximal linearly independent subset of V
6. S is a basis of V iff S is a minimal spanning set for V

Definition.

An *$m \times n$ matrix* over a field $\mathbb{F}$ is a function

$$A:\{1,2,\ldots,m\}\times\{1,2,\ldots,n\}\to \mathbb{F}.$$

A matrix is usually represented by (and identified with) an $m\times n$ ("m by n") array of elements of the field (usually numbers). The horizontal lines of a matrix are referred to as rows and the vertical ones as columns. The individual elements are called entries of the matrix.

Thus an $m\times n$ matrix has m rows, n columns and mn entries.

Matrices will be denoted by capital letters and their entries by the corresponding small letters. Thus, in case of a matrix A we will write $A(i,j)=a_{i,j}$ and will refer to $a_{i,j}$ as the element of the i-th row and j-th column of A.

On the other hand we will use the symbol $[a_{i,j}]$ to denote the matrix A with entries $a_{i,j}$. Rows and columns of a matrix can (and will) be considered vectors from $\mathbb{F}^n$ and $\mathbb{F}^m$, respectively, and will be denoted by $r_1, r_2, \ldots r_m$ and $c_1, c_2, \ldots, c_n$. The expression $m \times n$ is called the <u>size</u> of a matrix.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

Algebra of matrices

Definition.

Matrix addition is only defined for matrices of matching sizes,

(A+B)(i,j) = A(i,j)+B(i,j), $1\leq i\leq m$, $1\leq j\leq n$ (addition of functions).

(cA)(i,j) = cA(i,j), $1\leq i\leq m$, $1\leq j\leq n$ (multiplication of a function by a constant)

Fact.

The set of all m×n matrices over a field $\mathbb{F}$ ($\mathbb{F}^{m\times n}$) with these operations is a vector space over $\mathbb{F}$. Its dimension is mn.

Matrix multiplication. This is completely different story!

Definition.

Let A be an $m \times n$ and B a $p \times q$ matrix. The product AB is only defined if $n \times p$. Then

$$(AB)(i,j) = \sum_{s=1}^{n} A(i,s)\, B(s,j).$$

AB is clearly an $m \times q$ matrix.

Matrix multiplication is obviously noncommutative, it may happen that AB exists while BA does not.

Comprehension. Find an example of two 2×2 matrices A and B such that AB≠BA.

Matrix multiplication – example.

$$\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 0 & 3 \end{bmatrix} B$$

2 4 0

$$A \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 3 \end{bmatrix} A$$

$$B \begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 0 & 3 \end{bmatrix}$$

$$X \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

Definition.

Transposition is a unary operation on matrices. If A is an $m \times n$ matrix then "A transposed" is the $n \times m$ matrix A^T such that for each i and j ($1 \le i \le n, 1 \le j \le m$) $A^T(i,j) = A(j,i)$.

In other words, the first row of A becomes the first column of A^T and so on.

Definition.

If $A = A^T$ then A is said to be *symmetric*.

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,k} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,k} \end{bmatrix}^T = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{\mathrm{n},1} \\ a_{1,2} & a_{2,2} & \cdots & a_{n,2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{\mathrm{n},k} \end{bmatrix}$$

Example.

$$[1\ 3\ 4]^T = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix},$$

$$([1\ 3\ 4]^T)^T = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}^T = [1\ 3\ 4]$$

Fact. (obvious)

For every matrix A

$$(A^T)^T = A$$

Fact. (far less obvious but easy enough)
For every two matrices A and B such that AB exists

$$(AB)^T = B^T A^T$$

Proof.

$(AB)^T(\text{j}, \text{i})$ = (AB)(i,j) =

$\sum_{s=1}^{n}$ A(i,s) B(s,j) =

$\sum_{s=1}^{n}$ A^T(s,i) B^T(j,s) =

$\sum_{s=1}^{n}$ B^T(j,s) A^T(s,i) =

$B^T A^T$(j,i)

Switch to slide #15 of the old presentation